Two Transformations and $\ln(1+x)$

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1. Tools (s. [3])

Def.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a mapping.
We now define:

1. $\phi : J \to \mathbb{R}$ is convex, iff
   $\forall x, y \in J \quad \forall t \in [0; 1] \quad \phi(tx + (1 - t) y) \leq t\phi(x) + (1 - t) \phi(y)$

2. Let $\phi(J) \subseteq \mathbb{R}_+$.
   $\phi : J \to \mathbb{R}$ is logarithmically convex, iff
   $\ln(\phi) : J \to \mathbb{R}$ is convex.

Rem.: Let $\phi(J) \subseteq \mathbb{R}_+$.
Because $\exp : \mathbb{R} \to \mathbb{R}$ is convex and monotonically increasing, we get the following:

$(\phi : J \to \mathbb{R}$ is logarithmically convex$) \Rightarrow$
$(\phi : J \to \mathbb{R}$ is convex$)$

Theo.:

Pre.: Let J be a non-empty open interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a differentiable mapping.

Ass.: $(\phi : J \to \mathbb{R}$ is convex$) \Leftrightarrow$
$(\phi' : J \to \mathbb{R}$ is monotoncally increasing$)$

Theo.:

Pre.: Let J be a non-empty open interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a 2-times differentiable mapping.

Ass.: $(\phi : J \to \mathbb{R}$ is convex$) \Leftrightarrow$
$\phi'' \geq 0$

2. Gamma-Function (s. [3])

The Gamma-function $\Gamma : \mathbb{R}_+ \to \mathbb{R}$ is defined for all $\alpha \in \mathbb{R}_+$ through the absolutely convergent integral

$$\Gamma(\alpha) := \underbrace{\int_0^\infty \tau^{\alpha-1} \cdot e^{-\tau} \, d\tau}_{>0}$$

From literature we have:

$$\Gamma : \mathbb{R}_+ \to \mathbb{R} \text{ is analytically} \tag{1}$$

$$\forall \alpha \in \mathbb{R}_+ \quad \Gamma(\alpha + 1) = \alpha \cdot \Gamma(\alpha) \tag{2}$$

$$\forall k \in \mathbb{N}_0 \quad \Gamma(k + 1) = k! \tag{3}$$

$$\Gamma : \mathbb{R}_+ \to \mathbb{R} \text{ is logarithmically convex (and ergo convex)} \tag{4}$$

$$\Gamma(1) = 1 \text{ and } \Gamma(2) = 1 \tag{5}$$

With (4) and (5) we have:

$$\Gamma \mid [2; \infty[\text{ is monotonically increasing} \tag{6}$$

We now define a function $\gamma :]-1;\infty[\to \mathbb{R}$ through

$$\forall u \in]-1; \infty[\quad \gamma(u) := \Gamma(u + 1)$$

Then we have with (2):

$$\forall v \in]-1; \infty[\quad \gamma(v + 1) = (v + 1)\,\gamma(v) \tag{7}$$

In addition we have with (6):

$$\gamma \mid [1; \infty[\text{ is monotonically increasing} \tag{8}$$

3 Two Transformations for Power series

For every $\tilde{\alpha} \in \mathbb{R}_+$ there exists two transformations of power series:

$$\sum_{n=0}^{\infty} a_n x^n \quad \mapsto \quad \sum_{n=0}^{\infty} \frac{1}{n + \tilde{\alpha}} a_n x^{n+\tilde{\alpha}} \tag{T1}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{n!} b_n x^n \quad \mapsto \quad \sum_{n=0}^{\infty} \frac{1}{\gamma\left(n + \tilde{\alpha}\right)} b_n x^{n+\tilde{\alpha}} \tag{T2}$$

For every $t \in \mathbb{R}_+$, for wich the considered power series is absolutely convergent, the two transformations (T1) and (T2) are absolutely convergent too.

For the transformation (T1) we have:

$$\left(\sum_{n=0}^{\infty} \frac{1}{n + \tilde{\alpha}} a_n x^{n+\tilde{\alpha}}\right)' = x^{\tilde{\alpha}-1} \sum_{n=0}^{\infty} a_n x^n$$

Differentiating the transformation (T2) provides you ideally with a benign linear ODE. Then a comparison of the transformation (T1) with the terms of the explicit solution of the ODE (s. section 7) leads to an interesting equation. You can apply this method on every standard power series, for which there is a “good” inner connection of the b_n.

4 Take a Look At ln(*1+x*)

Let $\alpha \in \mathbb{R}_+$.
Let $J =]0;1[$.
We define a mapping $f : J \to \mathbb{R}$ through

$$\forall t \in J \quad f(t) := \ln(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} t^n$$

With Analysis we have:

$f : J \to \mathbb{R}$ is well-defined and differentiable

We now define the sequence $\left(b_n\right)_{n \in \mathbb{N}_0}$ through

$$\forall n \in \mathbb{N}_0 \quad b_n := \begin{cases} 0 & n = 0 \\ (-1)^{n-1}(n-1)! & n > 0 \end{cases}$$

Then the following is true:

$$\forall t \in J \quad f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} b_n t^n$$

5. Differentiating (T1)

We define a mapping $\tilde{f} : J \to \mathbb{R}$ through

$$\forall t \in J \quad \tilde{f}(t) := \frac{1}{1 - t} = \sum_{n=0}^{\infty} t^n$$

With Analysis we have:

$\tilde{f} : J \to \mathbb{R}$ is well-defined and differentiable

We now define the sequence $(a_n)_{n \in \mathbb{N}_0}$ through

$$\forall n \in \mathbb{N}_0 \quad a_n := 1$$

Then the following is true:

$$\forall t \in J \quad \tilde{f}(t) = \sum_{n=0}^{\infty} a_n t^n$$

Let $x = \left(\mathrm{id}_{\mathbb{R}}\right) \mid J$.

We now apply the transformation (T1) on $\tilde{f} : J \to \mathbb{R}$ and get the mapping $A_\alpha : J \to \mathbb{R}$:

$$\forall t \in J \quad A_\alpha(t) := \sum_{n=0}^{\infty} \frac{1}{n + (\alpha + 1)} a_n t^{n+(\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n + (\alpha + 1)} t^{n+(\alpha+1)} = \sum_{n=1}^{\infty} \frac{1}{n + \alpha} t^{n+\alpha}$$

Obviously $A_\alpha : J \to \mathbb{R}$ is well-defined and differentiable and the following is true:

$$\forall t \in J \quad A'_\alpha(t) = t^\alpha \cdot \tilde{f}(t) = \frac{t^\alpha}{1 - t}$$

that is

$$A_\alpha : J \to \mathbb{R} \text{ is an antiderivative of } \frac{x^\alpha}{1 - x} \text{ on } J \qquad (*)$$

6. Differentiating (T2)

We now apply the transformation (T2) on $f : J \to \mathbb{R}$ and get the mapping $B_\alpha : J \to \mathbb{R}$:

$$\forall t \in J \quad B_\alpha(t) := \sum_{n=0}^{\infty} \frac{b_n}{\gamma(n+\alpha)} t^{n+\alpha} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{\gamma(n+\alpha)} t^{n+\alpha}$$

Obviously $B_\alpha : J \to \mathbb{R}$ is well-defined and differentiable and it is for all $t \in J$:

$$\begin{aligned}
B'_\alpha(t) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{\gamma(n+\alpha)} (n+\alpha)\, t^{n+\alpha-1} = \\
&= \frac{\alpha+1}{\gamma(\alpha+1)} t^{\alpha} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(n-1)!}{\gamma(n+\alpha)} (n+\alpha)\, t^{n+\alpha-1} = \\
&= \frac{1}{\gamma(\alpha)} t^{\alpha} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(n-1)!}{\gamma(n+\alpha-1)} t^{n+\alpha-1} = \\
&= \frac{1}{\gamma(\alpha)} t^{\alpha} + \sum_{n=1}^{\infty} \frac{(-1)^{n}\, n!}{\gamma(n+\alpha)} t^{n+\alpha} = \\
&= \frac{1}{\gamma(\alpha)} t^{\alpha} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{\gamma(n+\alpha)} n t^{n+\alpha} = \\
&= \frac{1}{\gamma(\alpha)} t^{\alpha} - t^{\alpha+1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{\gamma(n+\alpha)} n t^{n-1} = \\
&= \frac{1}{\gamma(\alpha)} t^{\alpha} - t^{\alpha+1} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{\gamma(n+\alpha)} x^{n} \right)'(t) = \\
&= \frac{1}{\gamma(\alpha)} t^{\alpha} - t^{\alpha+1} \left(x^{-\alpha} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{\gamma(n+\alpha)} x^{n+\alpha} \right)'(t) = \\
&= \frac{1}{\gamma(\alpha)} t^{\alpha} - t^{\alpha+1} \left(x^{-\alpha} B_\alpha(x) \right)'(t)
\end{aligned}$$

For all $t \in J$ we have:

$$\left(x^{-\alpha} B_{\alpha}(x)\right)'(t) = -\alpha t^{-(\alpha+1)} B_{\alpha}(t) + t^{-\alpha} B'_{\alpha}(t)$$

Then we have for all $t \in J$:

$$\begin{aligned} B'_{\alpha}(t) &= \frac{1}{\gamma(\alpha)} t^{\alpha} - t^{\alpha+1}\left(-\alpha t^{-(\alpha+1)} B_{\alpha}(t) + t^{-\alpha} B'_{\alpha}(t)\right) = \\ &= \frac{1}{\gamma(\alpha)} t^{\alpha} + \alpha B_{\alpha}(t) - t B'_{\alpha}(t) \end{aligned}$$

that is

$$(1+t) B'_{\alpha}(t) = \frac{1}{\gamma(\alpha)} t^{\alpha} + \alpha B_{\alpha}(t)$$

that is

$$B'_{\alpha}(t) = \frac{1}{(1+t)\gamma(\alpha)} t^{\alpha} + \frac{\alpha}{1+t} B_{\alpha}(t)$$

Finally we come to the conclusion:

$$\begin{pmatrix} B_{\alpha} \text{ suffices the linear ODE} \\ \left(y_{\alpha}\right)' - \dfrac{\alpha}{1+x} y_{\alpha} = \dfrac{1}{(1+x)\gamma(\alpha)} x^{\alpha} \text{ on } J \end{pmatrix} \tag{9}$$

7. Solution of the ODE (s. [2])

Theo.:

Pre.: Let I be a non-empty open interval of $\mathbb{R}$.
Let $g : I \to \mathbb{R}$ be a continuous mapping.
Let $h : I \to \mathbb{R}$ be a continuous mapping.
Let $\xi \in I$.
Let $\eta \in \mathbb{R}$.

Ass.: The initial-value problem

$$y' + g(t)\, y = h(t) \qquad y(\xi) = \eta \qquad t \in I$$

has exactly one solution. It exists in all of I.

Rem.: Let $G : I \to \mathbb{R}$ be the antiderivative of $g : I \to \mathbb{R}$ with $G(\xi) = 0$, that is

$$\forall t \in I \qquad G(t) = \int_{\xi}^{t} g(\tau)\, d\tau$$

Then the solution of the initial-value problem is:

$$\forall t \in I \qquad y(t) = e^{-G(t)} \cdot \left(\eta + \int_{\xi}^{t} h(\tau) \cdot e^{G(\tau)}\, d\tau \right)$$

The integrals here are abviously well-defined.

8. Application of the Previous Theorem

For the application of the rule of substitution we need a mapping $\varphi : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{-1\}$, which is defined through

$$\forall s \in \mathbb{R} \setminus \{1\} \quad \varphi(s) := \frac{s}{1 - s}$$

Then the following is true:

$$\varphi : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{-1\} \text{ is differentiable}$$

$$\forall s \in \mathbb{R} \setminus \{1\} \quad \varphi'(s) := \frac{1}{(1 - s)^2}$$

$$\varphi\left(\left]0; \frac{1}{2}\right[\right) \subseteq \left]0; 1\right[$$

and

$$\varphi : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{-1\} \text{ is bijective}$$

$$\varphi^{-1} : \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{1\} \text{ is differentiable}$$

$$\forall t \in \mathbb{R} \setminus \{-1\} \quad \varphi^{-1}(t) := \frac{t}{1 + t}$$

With [3] the following theorem is true (Rule of Substitution):

Theo.:

Pre.: Let $s, s_0 \in \left]0; \frac{1}{2}\right[$ $\left(\text{Cave!: } \varphi\left(\left]0; \frac{1}{2}\right[\right) \subseteq \left]0; 1\right[\right)$.

Let the mapping $u : \left]0; 1\right[\to \mathbb{R}$ be defined through

$$\forall t \in \left]0; 1\right[\quad u(t) := \left(\frac{t}{1 + t}\right)^{\alpha} \frac{1}{1 + t}$$

Ass.: $u : \left]0; 1\right[\to \mathbb{R}$ is differentiable and it is:

$$\int_{\varphi(s_0)}^{\varphi(s)} u(\tau)\, d\tau = \int_{s_0}^{s} \sigma^{\alpha} \frac{1}{1 - \sigma}\, d\sigma$$

Rem.: The integrals here are obviously well-defined.

Proof: With [3] we have:

$$\int_{\varphi(s_0)}^{\varphi(s)} u(\tau)\, d\tau = \int_{s_0}^{s} u(\varphi(\sigma))\, \varphi'(\sigma)\, d\sigma$$

At that it follows for all $\sigma \in \left]0; \frac{1}{2}\right[$:

$$u(\varphi(\sigma))\, \varphi'(\sigma) = \left(\frac{\varphi(\sigma)}{1 + \varphi(\sigma)}\right)^{\alpha} \cdot \frac{1}{1 + \varphi(\sigma)} \cdot \frac{1}{(1 - \sigma)^2}$$

The terms can be interpreted $\left(\sigma \in \left]0; \frac{1}{2}\right[\right)$:

$$\left(\frac{\varphi(\sigma)}{1 + \varphi(\sigma)}\right)^{\alpha} = \left(\left(\varphi^{-1}\right)(\varphi(\sigma))\right)^{\alpha} = \sigma^{\alpha}$$

$$\frac{1}{1 + \varphi(\sigma)} = \frac{1}{1 + \frac{\sigma}{1 - \sigma}} = \frac{1}{\left(\frac{1}{1 - \sigma}\right)} = 1 - \sigma$$

The conclusion for all $\sigma \in \left]0; \frac{1}{2}\right[$ is:

$$u(\varphi(\sigma))\, \varphi'(\sigma) = \sigma^{\alpha} \cdot \frac{1}{(1 - \sigma)}$$

In the specific case of section 6. ist $I = J$ and the mappings $g : J \to \mathbb{R}$ and $h : J \to \mathbb{R}$ are defined through

$$\forall t \in J \quad g(t) := -\frac{\alpha}{1+t}$$

$$\forall t \in J \quad h(t) := \frac{1}{(1+t)\,\gamma(\alpha)}\, t^{\alpha}$$

We now define a mapping $T_\alpha : \left]0; \frac{1}{2}\right[\to \mathbb{R}$ through

$$\forall s \in \left]0; \frac{1}{2}\right[\qquad T_\alpha(s) := \gamma(\alpha) \cdot (1-s)^{\alpha} \cdot B_\alpha(\varphi(s))$$

We finally prove:

$$T_\alpha \text{ is an antiderivative of } \frac{x^{\alpha}}{1-x} \text{ on } \left]0; \frac{1}{2}\right[\qquad (**)$$

that is

$$\forall s, s_0 \in \left]0; \frac{1}{2}\right[\qquad T_\alpha(s) - T_\alpha(s_0) = \int_{s_0}^{s} \sigma^{\alpha} \cdot \frac{1}{1-\sigma}\, d\sigma$$

Proof:

Let $s_0 \in \left]0; \frac{1}{2}\right[$ and let $\xi := t_0 := \varphi(s_0) \in J$. Then we have for the antiderivative $G : J \to \mathbb{R}$ of $g : J \to \mathbb{R}$ with $G(\xi) = 0$:

$$\forall t \in J \quad G(t) = \int_{\xi}^{t} g(\tau)\, d\tau = -\alpha\left(\ln(1+t) - \ln(1+\xi)\right)$$

and

$$\forall t \in J \quad e^{G(t)} = \left(\frac{1+\xi}{1+t}\right)^{\alpha}$$

and

$$\forall t \in J \quad e^{-G(t)} = \left(\frac{1+t}{1+\xi}\right)^{\alpha}$$

Because B_α is a solution of the intial-value problem

$$y' + g(t)\, y = h(t) \qquad y(\xi) = B_\alpha(\xi) \qquad t \in J$$

We can apply the theorem of section 7. and get for all $t \in J$:

$$B_\alpha(t) = \left(\frac{1+t}{1+\xi}\right)^{\alpha} \left(B_\alpha(\xi) + \int_{\xi}^{t} \frac{\tau^\alpha}{(1+\tau)\,\gamma(\alpha)} \cdot \left(\frac{1+\xi}{1+\tau}\right)^{\alpha} d\tau\right) \tag{10}$$

At that equation we have for all $t \in J$:

$$\left(\frac{1 + t}{1 + \xi}\right)^{\alpha} B_{\alpha} (\xi) = \frac{(1 + t)^{\alpha}}{(1 + \xi)^{\alpha}} B_{\alpha} (\xi)$$

and

$$\left(\frac{1 + t}{1 + \xi}\right)^{\alpha} \cdot \int_{\xi}^{t} \frac{\tau^{\alpha}}{(1 + \tau) \gamma (\alpha)} \cdot \left(\frac{1 + \xi}{1 + \tau}\right)^{\alpha} d\tau =$$

$$\frac{(1 + t)^{\alpha}}{(1 + \xi)^{\alpha}} \cdot \int_{\xi}^{t} \frac{\tau^{\alpha}}{(1 + \tau) \gamma (\alpha)} \cdot \frac{(1 + \xi)^{\alpha}}{(1 + \tau)^{\alpha}} d\tau =$$

$$\frac{(1 + t)^{\alpha}}{\gamma (\alpha)} \cdot \int_{\xi}^{t} \frac{\tau^{\alpha}}{1 + \tau} \cdot (1 + \tau)^{-\alpha} d\tau =$$

$$\frac{(1 + t)^{\alpha}}{\gamma (\alpha)} \cdot \int_{\xi}^{t} \left(\frac{\tau}{1 + \tau}\right)^{\alpha} \cdot \frac{1}{1 + \tau} d\tau$$

Now we apply the subsitution $s = \dfrac{t}{1 + t}$ on the last two equations.

We get for all $s \in \left]0; \frac{1}{2}\right[$:

$$\left(\frac{1 + \varphi(s)}{1 + \xi}\right)^{\alpha} B_{\alpha}(\xi) = \frac{(1 + \varphi(s))^{\alpha}}{(1 + \xi)^{\alpha}} B_{\alpha}(\xi)$$

and

$$\left(\frac{1 + \varphi(s)}{1 + \xi}\right)^{\alpha} \cdot \int_{\xi}^{\varphi(s)} \frac{\tau^{\alpha}}{(1 + \tau)\,\gamma(\alpha)} \cdot \left(\frac{1 + \xi}{1 + \tau}\right)^{\alpha} d\tau =$$

$$\frac{(1 + \varphi(s))^{\alpha}}{\gamma(\alpha)} \cdot \int_{\varphi(s_0)}^{\varphi(s)} \left(\frac{\tau}{1 + \tau}\right)^{\alpha} \cdot \frac{1}{1 + \tau}\, d\tau =$$

$$\frac{(1 + \varphi(s))^{\alpha}}{\gamma(\alpha)} \cdot \int_{s_0}^{s} \sigma^{\alpha} \cdot \frac{1}{1 - \sigma}\, d\sigma$$

Now we can interpret the terms in (10) and get for all $s \in \left]0; \frac{1}{2}\right[$:

$$B_\alpha\left(\varphi(s)\right) = \frac{\left(1 + \varphi(s)\right)^\alpha}{\left(1 + \varphi\left(s_0\right)\right)^\alpha} B_\alpha\left(\varphi\left(s_0\right)\right) + \\ + \frac{\left(1 + \varphi(s)\right)^\alpha}{\gamma(\alpha)} \cdot \int_{s_0}^{s} \sigma^\alpha \cdot \frac{1}{1 - \sigma}\, d\sigma \qquad (11)$$

that is

$$\gamma(\alpha)\left(\left(1 + \varphi(s)\right)^{-\alpha} B_\alpha\left(\varphi(s)\right) - \left(1 + \varphi\left(s_0\right)\right)^{-\alpha} B_\alpha\left(\varphi\left(s_0\right)\right)\right) = \\ = \int_{s_0}^{s} \sigma^\alpha \cdot \frac{1}{1 - \sigma}\, d\sigma$$

that is $\left(\forall \tilde{s} \in \left]0; \frac{1}{2}\right[\quad 1 + \varphi\left(\tilde{s}\right) = \frac{1}{1 - \tilde{s}}\right)$

$$T_\alpha(s) - T_\alpha\left(s_0\right) = \int_{s_0}^{s} \sigma^\alpha \cdot \frac{1}{1 - \sigma}\, d\sigma$$

9. An Interesting Equation

With (*) and (**) we have:

$$\forall \alpha \in \mathbb{R}_+ \quad \left(A'_\alpha - T'_\alpha = 0 \text{ on } \left]0; \frac{1}{2}\right[\right)$$

So there exists a mapping $\lambda : \mathbb{R}_+ \to \mathbb{R}$ with

$$\forall \alpha \in \mathbb{R}_+ \quad \forall s \in \left]0; \frac{1}{2}\right[\quad \lambda(\alpha) = A_\alpha(s) - T_\alpha(s)$$

Because $\forall \alpha \in \mathbb{R}_+ \quad \lim_{s \to 0+} A_\alpha(s) = \lim_{s \to 0+} T_\alpha(s) = 0$ it follows:

$$\forall \alpha \in \mathbb{R}_+ \quad \lambda(\alpha) = 0$$

With the definitions of A_α and T_α we have for all $\alpha \in \mathbb{R}_+$ and for all $s \in \left]0; \frac{1}{2}\right[$:

$$\sum_{n=1}^{\infty} \frac{1}{n + \alpha} s^{n+\alpha} - \gamma(\alpha)(1 - s)^\alpha \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{\Gamma(n + \alpha + 1)} (\varphi(s))^{n+\alpha} = 0$$

At that we can interpret the terms:

$$\forall n \in \mathbb{N}_+ \quad \forall \alpha \in \mathbb{R}_+ \quad \frac{\gamma(\alpha)}{\Gamma(n + \alpha + 1)} = \frac{1}{n + \alpha} \cdot \prod_{i=1}^{n-1} \frac{1}{i + \alpha}$$

and

$$\forall n \in \mathbb{N}_0 \quad \forall \alpha \in \mathbb{R}_+ \quad \forall s \in \left]0; \frac{1}{2}\right[\quad (1 - s)^\alpha (\varphi(s))^{n+\alpha} = \frac{s^{n+\alpha}}{(1 - s)^n}$$

Finally we come to the following conclusion (for all $\alpha \in \mathbb{R}_+$ and all $s \in \left]0; \frac{1}{2}\right[$):

$$\sum_{n=1}^{\infty} \left(\left(\prod_{i=1}^{n-1} \frac{1}{i} \right) - \frac{(-1)^{n-1}}{(1 - s)^n} \left(\prod_{i=1}^{n-1} \frac{1}{i + \alpha} \right) \right) \frac{(n - 1)!}{n + \alpha} s^{n+\alpha} = 0$$

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